Generation of Mean Flows in a Rotating Convection Layer

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To Professor Arnulf Schlüter on his 60th Birthday

The instability of convection rolls in a fluid layer heatet from below is studied in the case where the layer rotates about an axis slightly inclined with respect to the vertical. The inclination destroys the horizontal isotropy of the layer, but the instability of rolls found by Küppers and Lortz [1] is little affected as long as the angle of inclination is small. A new effect is the generation of mean Reynolds stresses by rolls not aligned with the horizontal component of the rotation vector. The mean flow exhibits a vorticity of the same sign as the horizontal component of rotation and agrees qualitatively with the mean flow found in the numerical experiments of Hathaway and Somerville [2].

1. Introduction

Thermal convection under the influence of rotation has been recognized in recent years as the origin of dynamical features observed on the sun and in the atmospheres of the major planets. Convection driven by thermal or chemical buoyancy is also considered the cause of the generation of magnetic fields in the cases of rapidly rotating planets such as Earth, Jupiter and Saturn. Among the most striking features associated with convection in rotating spherical shells is the generation of mean zonal flows. Simple analytical models (Busse [3]), laboratory experiments (Busse and Hood [4]) and three-dimensional numerical computations (Gilman [5]) have demonstrated that the nonlinear advection of momentum by the fluctuating convection motion leads to the generation of zonal flows. Common to all these models is the Rossby wave like character of the convection modes and the important role played by the variation of the dimension of the fluid system in the direction of the axis of rotation as a function of the distance from that axis.

For this reason it was surprising when Hathaway and Somerville [2] recently found a generation of mean flows by convection in a layer of constant height. Their layer differed from the traditionally studied case in that the vector of rotation possesses a horizontal component. Without this component the layer is horizontally isotropic provided that the effect of the centrifugal force can be neglected. Thus a mean flow requiring a preferred horizontal direction can not be expected. Even when the axis of rotation is inclined, a mean flow is surprising. For convection rolls aligned with the horizontal component of the rotation vector, the corresponding Coriolis force is balanced entirely by the pressure and the dynamics of the problem becomes identical to that with a purely vertical axis of rotation. No mean flow can thus be generated by the aligned convection rolls which are expected to be the preferred form of convection according to linear analyses (Chandrasekhar [6]).

In this paper it is shown that the generation of a mean flow such as that found by Hathaway and Somerville is caused by an instability first discovered by Küppers and Lortz [1]. In the case of a vertical axis of rotation, this instability leads to a state of weakly nonlinear turbulence associated with the cyclically changing directions of the convection rolls. In the case of an inclined axis of rotation the novel phenomenon of a mean flow appears. While experimental observations are available in the case of a vertical axis of rotation (Busse and Heikes [7]; Heikes and Busse [8]) no laboratory evidence exists yet for the case of horizontal component of rotation. Such an experiment could easily be arranged by using a parabolically shaped layer which would exhibit the additional advantage that the effective gravity including the centrifugal force is normal to the isotherm of the basic conductive state for an appropriate rotation rate. This kind of experiment has been contemplated in order to avoid the com-

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plications of the centrifugally induced circulation in a plane layer. The problem is of geophysical and astrophysical importance in that it models convection in rotating thin spherical fluid shells outside the equatorial regime. It may have applications in the polar regions of Jupiter where convection is likely to be distinctly different from that observed in the low latitudes of the planets (Busse [9]). These applications will not be considered in this paper which is focussed on the mechanism of mean flow generation.

After formulating the basic mathematical problem in Sect. 2, linear aspects are discussed in Sect. 3. The modifications of the Küppers-Lortz instability owing to the horizontal component of the rotation vector are considered in Section 4. The mean flow problem is attacked in Sect. 5 and a few remarks on more general implications of the problem are added in a concluding section.

2. Mathematical Formulation

We consider a horizontal layer of fluid of thickness d which is stationary with respect to a system rotating with the constant angular velocity $\boldsymbol{\Omega} = \Omega_v \boldsymbol{k} + \Omega_h \boldsymbol{j}$. The unit vector $\boldsymbol{k}$ is in the direction opposite to gravity while $\boldsymbol{j}$ is a horizontal unit vector which is sometimes identified with the northward direction in reference to potential geophysical applications of the problem as shown in Figure 1.

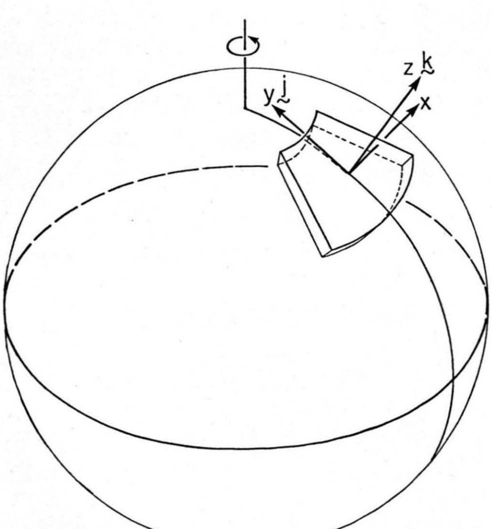

Fig. 1. Sketch of the convection layer with inclined axis of rotation in a geophysical context.

The temperatures T_2 and T_1 are prescribed at the lower and upper boundaries of the layer. Using d, $d^2/\varkappa$, and $(T_2 - T_1) R^{-1}$ as scales for length, time, and temperature, respectively, the basic equations for the velocity vector $\boldsymbol{u}$ and the deviation θ of the temperature from the static distribution can be written in the form,

$$P^{-1}(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -\nabla \pi + \theta \boldsymbol{k} + \nabla^2 \boldsymbol{u} - (\tau \boldsymbol{k} + \eta \boldsymbol{j}) \times \boldsymbol{u}, \quad (1\text{a})$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad (1\text{b})$$

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = R \boldsymbol{u} \cdot \boldsymbol{k} + \nabla^2 \theta. \quad (1\text{c})$$

The symbol ∂_t indicates the differentiation with respect to time. The equations include four dimensionless parameters,

Rayleigh number $R = \gamma g (T_2 - T_1) d^3/\nu\varkappa$,
Prandtl number $P = \nu/\varkappa$,
vertical rotation parameter $\tau = 2\Omega_v d^2/\nu$,
horizontal rotation parameter $\eta = 2\Omega_h d^2/\nu$,

where γ, ν, and $\varkappa$ are coefficients of thermal expansion, kinematic viscosity and thermal diffusivity, respectively. The acceleration of gravity is denoted by g. Equation (16) can be eliminated by using the general representation for the solenoidal vector field $\boldsymbol{u}$,

$$\boldsymbol{u} = \nabla \times (\nabla \times \boldsymbol{k} v) + \nabla \times \boldsymbol{k} w. \quad (2)$$

By operating with $\boldsymbol{k} \cdot \nabla \times (\nabla \times \ldots)$ and $\boldsymbol{k} \cdot \nabla \times$ onto Eq. (1a) two scalar equations for v and w are obtained,

$$\nabla^4 \Delta_2 v - \Delta_2 \theta - (\tau \boldsymbol{k} + \eta \boldsymbol{j}) \cdot \nabla \Delta_2 w = (\boldsymbol{k} \cdot \nabla \times (\nabla \times \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \partial_t \nabla^2 \Delta_2 v) P^{-1}, \quad (3\text{a})$$

$$\nabla^2 \Delta_2 w + (\tau \boldsymbol{k} + \eta \boldsymbol{j}) \cdot \nabla \Delta_2 v = (\boldsymbol{k} \cdot \nabla \times (\boldsymbol{u} \cdot \nabla u) + \partial_t \Delta_2 w) P^{-1}, \quad (3\text{b})$$

$$\nabla^2 \theta - R \Delta_2 v = \boldsymbol{u} \cdot \nabla \theta + \partial_t \theta, \quad (3\text{c})$$

where the operator Δ_2 is defined by

$$\Delta_2 \equiv \nabla^2 - (\boldsymbol{k} \cdot \nabla)^2.$$

The heat Eq. (3c) has been added to complete the equations describing the problem. For simplicity we shall restrict the attention to the case of stress free boundaries,

$$v = \partial^2_{zz} v = \partial_z w = \partial = 0 \quad \text{at} \quad z = \pm \tfrac{1}{2}. \quad (4)$$

Here we have used a Cartesian system of coordinates with the z-coordinate in the direction of $\boldsymbol{k}$ and the origin on the median plane of the layer. The y-coordinate will be assumed in the direction of $\boldsymbol{j}$. In the next section the linear part of (3) will be solved in two limits. First the limit of small η and arbitrary τ will be considered. Then the case of large τ and arbitrary η will be analyzed.

3. The Linear Problem

Chandrasekhar [6] has shown that the Rayleigh number for onset of convection corresponds to steady infinitesimal solutions of (3) when the Prandtl number exceeds a value of about 0.68. Restricting our attention to this case we consider (3) in the limit when the right hand sides can be neglected.

By eliminating w and θ the following linear equation for v is obtained

$$[\nabla^6 - R\Delta_2 + (\tau\,\partial_z + \eta\,\partial_y)^2]\,v = 0\,. \tag{5}$$

An explicit analytical solution of this equation is cumbersome. A considerable simplification is obtained when the limit of small η is assumed. In this case the solution v, R can be written in the form of a power series in η,

$$\begin{aligned} v &= v_0 + \eta v_1 + \eta^2 v_2 + \cdots, \\ R &= R_0 + \eta R_1 + \eta^2 R_2 + \cdots. \end{aligned} \tag{6}$$

In lowest order the problem reduces to the problem with vertical axis of rotation (Chandrasekhar [6]) which yields the solution

$$v_0 = \cos\pi z \exp[i\alpha x + i\beta y]\,, \tag{7a}$$

$$R_0 = [(\pi^2 + a^2)^3 + (\tau\pi)^2]\,a^{-2}\,, \tag{7b}$$

$$a^2 \equiv \alpha^2 + \beta^2\,.$$

Of physical interest is the minimum R_{0c} of R_0 as a function of a. This minimum increases proportionally to $\tau^{4/3}$ asymptotically while the corresponding value a_c increase with $\tau^{1/3}$. But in the following analysis it is not necessary to specify the value a_c.

The equation of first order in η is

$$\begin{aligned} &[\nabla^6 - R_0\Delta_2 + \tau^2\,\partial_{zz}^2]\,v_1 \\ &= -2\tau\,\partial_{zy}^2 v_0 + R_1\Delta_2 v_0\,. \end{aligned} \tag{8}$$

Since the operator on the left hand side is self adjoint and its homogeneous solutions are symmetric in z, the solvability condition for the inhomogeneous Eq. (8) is satisfied with $R_1 = 0$. By expanding the right hand side in terms of the functions $\sin 2n\pi z$ which satisfy the boundary conditions (4) the solution v_1 can be obtained in the form

$$\begin{aligned} v_1 = {} & i\,16\,\beta\tau \sum_{n=1}^{\infty} (-1)^n D_n n (4n^2 - 1)^{-1} \\ & \cdot \sin 2n\pi z \exp[i\alpha x + i\beta y]\,, \end{aligned} \tag{9}$$

where the definition

$$D_n = \{[(2n\pi)^2 + a^2]^3 - R_0 a^2 + (2n\pi\tau)^2\}^{-1} \tag{10}$$

has been used.

In second order equation (5) yields

$$\begin{aligned} &[\nabla^6 - R_0\Delta_2 + \tau^2\,\partial_{zz}^2]\,v_2 \\ &= -2\tau\,\partial_{zy}^2 v_1 - \partial_{yy}^2 v_0 + R_2\Delta_2 v_0\,. \end{aligned} \tag{11}$$

The solvability condition that the right hand side be orthogonal to v_0^+ yields the following expression for R_2

$$\begin{aligned} R_2 a^2 &= \beta^2 \Big(1 - \sum_{n=1}^{\infty} 16 D_n (4n)^2 (4n^2 - 1)^{-2}\Big) \\ &\equiv \beta^2 a^2 Q\,. \end{aligned} \tag{12}$$

An inspection of the infinite sum reveals that it is a monotonically decreasing function of τ if the critical value of a is used. In the limit $\tau \to \infty$ it approaches unity as is shown below by a different analysis. Thus for finite τ, Q is positive which confirms the expectation (Chandrasekhar [6]) that convection modes with finite β require a higher Rayleigh number than convection rolls aligned with the axis of rotation corresponding to $\beta = 0$. For rigid boundaries this property has been found in the numerical analysis of Hathaway et al. [10]. For later use it is necessary to obtain the solution for w. Assuming an expansion analogous to (6) we find from (3b) with vanishing right hand side

$$\begin{aligned} w_0 = {} & -\pi\tau \sin\pi z (\pi^2 + a^2)^{-1} \\ & \cdot \exp\{i\alpha x + i\beta y\}\,, \end{aligned} \tag{13a}$$

$$\begin{aligned} w_1 = {} & \frac{2i}{\pi} \exp\{i\alpha x + i\beta y\} \\ & \cdot \sum_{n=0}^{\infty} (-1)^n (4n^2 - 1)^{-1} [(2n\pi)^2 + a^2]^{-1} \\ & \cdot [(4n\pi)^2 \tau^2 D_n - (2 - \delta_{n0})] \\ & \cdot \cos 2n\pi z\,. \end{aligned} \tag{13b}$$

An alternative solution of (5) in the limit of large values of τ can be obtained from the equation

$$[a^6 - R a^2 - (\tau\,\partial_z + \eta\,\partial_y)^2]\,v = 0\,, \tag{14}$$

since $a^2 \gg \pi^2$ holds in that limit for the preferred value of a. The solution of this equation satisfying the boundary condition $v=0$ at $z=\pm\frac{1}{2}$ is given by

$$v = \sin n\pi(z-\tfrac{1}{2}) \cdot \exp\{i(\alpha x + \beta y - \beta\eta z/\tau)\} \quad (15)$$

corresponding to

$$R = [a^6 + (n\pi\tau)^2]/a^2 . \quad (16)$$

Obviously the lowest value of R is reached for $n=1$. It is remarkable that in the order to which the problem has been considered the Rayleigh number (16) is independent of η. Higher order contributions to R dependent on η arise from boundary layers which are needed to satisfy the remaining boundary conditions (4) not fulfilled by solution (15). The vanishing dependence of R on η in lowest order becomes particularly striking in the limit $\eta \gg \tau$. In this case the solution (15) describes flows in the form of sheets parallel to the direction $\eta\boldsymbol{j}+\tau\boldsymbol{k}$. The nearly vanishing dependence on this direction allows the flow to conform approximately to the Proudman-Taylor theorem which states that small amplitude steady flows must be independent of the coordinate in the direction of the axis of rotation in the limit $(\eta^2+\tau^2)^{1/2} \to \infty$.

4. The Nonlinear Problem

For the remainder of this paper we shall return to the assumption of small η. This allows us to use the analysis of Küppers and Lortz [1] with only small modifications. Their paper will be referred to by KL in the following. In extending the work of Schlüter et al. [11] to the case of a fluid layer rotating about a vertical axis, Küppers and Lortz found that convection in the form of rolls is the only stable steady solution for $\tau < \tau_c$. For $\tau > \tau_c$ an instability occurs which prevents any steady solution from being physically realized. The instability has the form of rolls inclined with an angle of about 60° with respect to the direction of the original rolls.

Following KL we start with a general solution of the linear problem

$$v^{(0)} = \sum_{n=-N}^{N} C_n(t) \exp\{i\boldsymbol{l}_n \cdot \boldsymbol{r}\} \cos\pi z , \quad (17)$$

where $\boldsymbol{r}$ is the position vector and the vectors $\boldsymbol{l}_n$ and coefficients $C_n(t)$ satisfy the conditions

$$\boldsymbol{l}_n \cdot \boldsymbol{k} = 0 , \quad \boldsymbol{l}_{-n} = -\boldsymbol{l}_n , \quad C_{-n} = C_n^+ .$$

C_n^+ denotes the complex conjugate of C_n. In contrast to KL a time dependence of the coefficients C_n has been admitted in order to permit the analysis of time dependent problems. The amplitude ε of solution (17) is regarded as small,

$$\varepsilon \equiv \sum_{n=-N}^{N} |C_n|^2 \ll 1$$

such that the nonlinear terms in Eqs. (3) may be treated as perturbations. The analysis of the nonlinear terms then leads to a system of equations of the form

$$\begin{aligned} &\tfrac{1}{2} M (\mathrm{d}/\mathrm{d}t) |C_i(t)|^2 \\ &= \{[R - R_{0c} - (\boldsymbol{l}_i \cdot \boldsymbol{j})^2 \eta^2 Q] K \\ &\qquad - \tfrac{1}{2} \sum_{m=-N}^{N} T_{im} |C_m|^2\} |C_i|^2 \\ &\text{for} \quad i = 1, \ldots, N . \end{aligned} \quad (18)$$

This system differs from the corresponding system (7.9) of KL in two respects. First a weak time dependence of the order ε^2 is taken into account and yields the additional term on the left hand side. Secondly, the variation of the critical Rayleigh number owing to a small, but finite value of η is taken into account and appears in the form of the parameter Q (see definition (12)) in (18). The approximation on which the derivation of (18) is based requires that η and ε are of the same order.

As has been shown in earlier work (Busse and Clever [12]; Busse and Heikes [7]) the weakly nonlinear turbulence which results from the Küppers-Lortz instability can be described locally by a reduced system of the form (18) with $N=3$ and

$$\boldsymbol{l}_1 + \boldsymbol{l}_2 + \boldsymbol{l}_3 = 0 . \quad (19)$$

For $\eta \ll \varepsilon$ the statistical limit cycle exhibited by the reduced system (Busse [13]) is little changed from the case $\eta = 0$. As η increases the mode corresponding to a minimum value of $|\boldsymbol{l}_n \cdot \boldsymbol{j}|$ predominates and the solution is represented by that mode for a larger part of the cycle than by each of the two other modes. But basically the cyclical character of the time dependent solution remains unchanged. Only when η exceeds a finite value of the order ε, does the Küppers-Lortz instability disappear. In other words, for finite values of η there always exists a region in the neighborhood of $R = R_{0c}$ where steady state convection rolls aligned with $\boldsymbol{j}$ are stable for arbitrary values of τ. When $R - R_{0c}$ exceeds a finite value of the order $\eta^2 Q$ the Küppers-Lortz instability occurs and the time dependence of convection can be described by the statistical limit cycle behavior. Here the quantitative aspects of the problem will not be studied in more detail. Instead we turn to a novel feature of the problem with finite η.

5. Generation of Mean Flow

By taking the average over the x- and y-coordinates of the horizontal component of Eq. (1a) the following equation is obtained

$$\partial_{zz}^2 \overline{\boldsymbol{u}} = P^{-1} \partial_z \overline{\Delta_2 v (\boldsymbol{k} \times \nabla w - \boldsymbol{k} \cdot \nabla \nabla v)} + \tau \boldsymbol{k} \times \overline{\boldsymbol{u}}, \tag{20}$$

where the bar indicates the horizontal average. We first evaluate the Reynolds stress for a convection mode described by (6), (7a). Note that only the real parts of (7a), (9), (13) enter into the analysis;

$$\begin{aligned}\overline{\Delta_2 v (\boldsymbol{k} \times \nabla w - \boldsymbol{k} \cdot \nabla \nabla v)} &= a^2 \eta \{\boldsymbol{j} \times \boldsymbol{k} \overline{(v_0 \partial_y w_1 + v_1 \partial_y w_0)} + \boldsymbol{j} \overline{(v_0 \partial_{zy}^2 v_1 + v_1 \partial_{zy}^2 v_0)}\} \\ &= a^2 \eta \beta^2 \sum_{n=0}^{\infty} \{\boldsymbol{j} \times \boldsymbol{k} S_n + \boldsymbol{j} T_n\} \cos(2n+1)\pi z,\end{aligned} \tag{21}$$

where the constants S_n and T_n are defined by

$$\begin{aligned} S_n \equiv (-1)^n \{&(4n^2 - 1)^{-1} [(1 - 8n^2 \tau^2 D_n)(a^2 + 4n^2\pi^2)^{-1} + 4n\tau^2\pi^2 D_n (\pi^2 + a^2)^{-1}] \\ &- (4(n+1)^2 - 1)^{-1} [(1 - 8(n+1)^2 \tau^2 \pi^2 D_{n+1})(a^2 + 4(n+1)^2\pi^2)^{-1} \\ &- 4(n+1) D_{n+1} \tau^2 \pi^2 (\pi^2 + a^2)^{-1}]\}/\pi, \end{aligned} \tag{22a}$$

$$T_n = (-1)^n 4\pi\tau[(n+1)D_{n+1} - nD_n]/(2n+1). \tag{22b}$$

According to (21) the Reynolds stress does not depend on the sign of β. If we thus evaluate the Reynolds stress for the triad of rolls involved in the statistical limit cycle induced by the Küppers-Lortz instability, we obtain

$$\begin{aligned}&\overline{\Delta_2 v^* (\boldsymbol{k} \times \nabla w^* - \boldsymbol{k} \cdot \nabla \nabla v^*)} \\ &= a^2 \eta \beta^2 4 (|C_2|^2 + |C_3|^2) \\ &\cdot \sum_{n=0}^{\infty} (\boldsymbol{j} \times \boldsymbol{k} S_n + \boldsymbol{j} T_n) \cos(2n+1)\pi z.\end{aligned} \tag{23}$$

We have assumed that the component of order η^0 of v^* is given by (17) with $N = 3$ and that three vectors of relationship (19) are oriented such that $\boldsymbol{l}_1$ is perpendicular to $\boldsymbol{j}$. This is certainly a good assumption for an experiment started from conditions at which stable convection rolls with $\boldsymbol{l}_1 \cdot \boldsymbol{j} = 0$ exist. Accordingly

$$\beta \approx \sqrt{3} a/2 \tag{24}$$

holds. In the limit $\eta \to 0$ the average in time of $|C_n(t)|^2$ can be expressed in terms of $R - R_{0c}$ using expressions of Veronis [14]

$$2|C_n|^2 \approx \tfrac{1}{3}\varepsilon^2 = E(R - R_{0c})/4R_{0c} \quad \text{for} \quad n = 1, 2, 3, \tag{25}$$

where the constant E is defined by

$$E = 16(\pi^2 + a^2)/(1 - P^2\pi^4\tau^2/a^2 R_{0c}).$$

The time average is equivalent to a spatial average in the rotating layer since the phase of the statistical limit cycle varies from one patch of convection rolls to the next as demonstrated by the experimental observations (Busse and Heikes [7]; Heikes and Busse [8]).

After introducing the Reynolds stress (23), (20) can most easily be integrated by defining the complex variable

$$U \equiv \bar{u}_x + i\bar{u}_y \tag{26}$$

which allows us to rewrite (20) in the form

$$\begin{aligned}(\partial_{zz}^2 - i\tau) U = &-\eta\beta^2 P^{-1}(R - R_{0c}) E/R_{0c} \\ &\cdot \sum_{n=0}^{\infty} (2n+1)\pi(S_n + iT_n) \\ &\cdot \sin 2(n+1)\pi z.\end{aligned} \tag{27}$$

Integration of (27) yields the following expressions for $\bar{u}_x$ and $\bar{u}_y$

$$\begin{aligned}\bar{u}_x = &\eta\beta^2(R - R_{0c}) E P^{-1} \sum_{n=0}^{\infty} (2n+1)\pi \\ &\cdot [(2n+1)^2\pi^2 S_n + \tau T_n] \\ &\cdot [(2n+1)^4\pi^4 + \tau^2]^{-1} \sin(2n+1)\pi z,\end{aligned} \tag{28a}$$

$$\bar{u}_y = -\eta\beta^2(R - R_{0c})EP^{-1}\sum_{n=0}^{\infty}(2n+1) \cdot \pi(\tau S_n - (2n+1)^2\pi^2 T_n) \cdot [(2n+1)^4\pi^4 + \tau^2]^{-1}\sin(2n+1)\pi z \cdot \quad (28\text{b})$$

Obviously this solution satisfies the stress free boundary conditions at $z = \pm\frac{1}{2}$.

To get an impression of the form of the solution it is sufficient to evaluate the term with $n = 0$ in the summation which is by far the largest. Since S_0 is negative, and T_0 is quite small

$$\pi S_0 = -a^{-2} - [(1 - D_1\tau^2 8\pi^2) \cdot (a^2 + 4\pi^2)^{-1} - 4D_1\tau\pi^2 \cdot (a^2 + \pi^2)^{-1}]/3, \quad (29\text{a})$$

$$T_0 = 4\tau\pi D_1.$$

$\bar{\boldsymbol{u}}$ is pointed in the north-western direction in the upper half of the layer. For $z < 0$ the direction is reversed. This feature agrees with the findings of Hathaway and Somerville [2]. Even the amplitude agrees roughly if expressed in terms of $(R - R_{0c})/R_{0c}$. But a good quantitative comparison cannot be expected because of the rigid boundaries assumed by Hathaway and Somerville.

6. Concluding Remarks

In studying physical phenomena theoretical physicists tend to focus their attention on the configuration of highest symmetry which exhibits the phenomenon of interest. Sometimes, however, small asymmetries may give rise to novel features. Perhaps the preoccupation with the most symmetric configurations has prevented fluid dynamicists from discovering earlier the phenomenon first found by Hathaway and Somerville [2] and studied analytically in this paper. The horizontal layer with inclined axis of rotation is not the only configuration in which the process of mean flow generation by convection can be investigated; but it is certainly the most accessible case from a mathematical point of view. It thus deserves further study apart from its potential applications in geophysics and astrophysics.

In the present paper the Küppers-Lortz instability has been emphasized as the mechanism by which deviations are caused from the steady solution in the form of rolls aligned with the northward direction. Any other three dimensional instability, however will yield the same result. The skewed varicose instability and the oscillatory instability preceed the Küppers-Lortz instability at sufficiently low values of τ (Clever and Busse [15]). At finite amplitudes those instabilities will produce a mean flow of similar form as that considered in this paper provided η is finite.

It is remarkable that the vorticity of $\bar{u}_x$ has the same sign as the component Ω_h of the rotation vector. This property contradicts the superficial notion, that convection tends to mix angular momentum. In fact the angular momentum contrast is increased by a finite $\bar{u}_x$ beyond its value in the static state.

The generation of Reynolds stresses can be traced to the inclination of the boundary between convection rolls which are not oriented in the north-south direction. This inclination is evident both from the asymptotic form (15) of the solution and from the combined expressions (7a) and (9). Because of the northward tilt a finite upward advection of north-western momentum occurs which is balanced by the opposite viscous stress of the mean flow. Because of the close analogy of the effects of magnetic fields and rotation on the onset of convection the question arises whether similar mean flows are generated in a fluid layer obliquely intersected by a homogeneous magnetic field. The answer to this question is not obvious since an instability of the Küppers-Lortz type does not occur in a non-rotating layer in the presence of a magnetic field (Busse and Clever [16]). Other instabilities of convection rolls aligned with the horizontal component of the magnetic field are inhibited. A numerical analysis will be necessary to answer the question because of the relatively large amplitudes of convection at which three-dimensional convection can be expected.

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